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EDGE CORRECTIONS IN THE CALCULATION OF THE ABSOLUTE CAPACITY OF CONDENSERS BY THE SCHWARZIAN TRANSFORMATION.

BY J. G. COFFIN.

Presented by A. G. Webster, December 9, 1903. Received December 11, 1903.

IN the construction of an absolute condenser, consisting of two silvered plane circular glass plates, it was found necessary to compute the edge corrections for many cases which have not yet been determined.

In the following pages will be found a number of such cases worked out by means of the general conformal transformation due to Cristoffel * and Schwarz.†

A very clear résumé of the method will be found in the chapter on conjugate functions of J. J. Thomson's Recent Researches in Electricity and Magnetism.

The method has been applied to problems of different kinds by Kirchhoff,‡ Potier,§ Michell,‖ and Love.¶

It will be noticed that in all the cases here presented, there is a variable parameter which allows the correction for any relative position or size of the edges to be given from the general solution.

It then follows that a number of cases already solved by J. J. Thomson are but special cases of these the more general solutions.

This is a great advantage, as it affords an excellent check upon the accuracy of the more complicated results.

In a number of cases the problem was solved in two distinct manners, the more elegant being here presented, but the results in every case agreeing with each other.

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* Annali de Math., **1**, 89 (1867).

† Crelle, **70**, 105–120 (1869).

‡ Gesam. Abhand., p. 101.

§ French Trans. of Maxwell, Appendix.

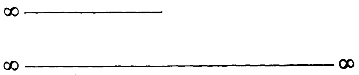
‖ Phil. Trans., A, 389 (1890).

¶ Proc. Camb. Phil. Soc., **7**, 175 (1891).

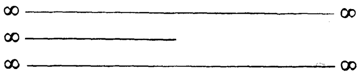
Professor J. J. Thomson solves the following problems, which are most concisely stated by diagrams. The sign ∞ at the end of a line, denotes that the line is to be considered to extend to infinity in that direction.

Every line represents the intersection with the plane of the paper of planes perpendicular to the paper and extending to infinity in both directions, above and below. So that figure (a), for example, represents a semi-infinite plane, near and parallel to another infinite plane.

In all the diagrams these planes are to be considered conductors of electricity.



(a)



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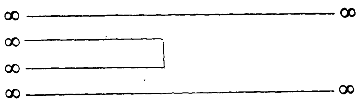

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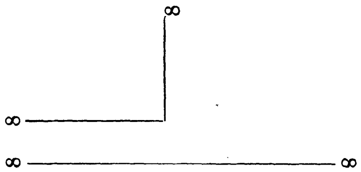

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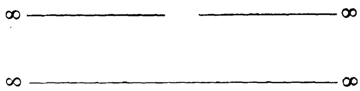

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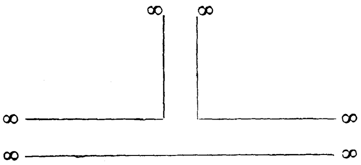
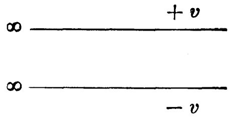


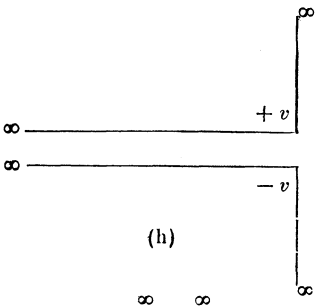
(f)

The last two cases have special reference to the correction for the slit in a guard ring electrometer.

For further work on the electrometer from this point of view see the French translation of Maxwell, Vol. II, p. 568, et seq., where M. Potier has worked out more general cases. We can immediately extend these results of Thomson's by the method of electrical images. We thus obtain the solutions for the following cases, where v represents the potential:



(g)



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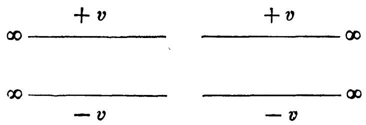

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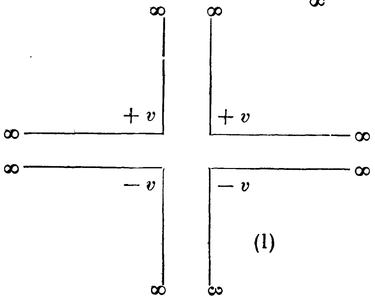
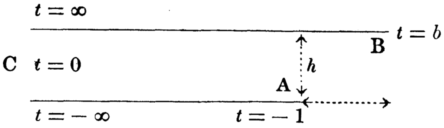


(l)

Problem I.

Consider two semi-infinite planes one of whose edges extends beyond the other.

At A $t_1 = -1$, $\alpha_1 = 2\pi$
C $t_2 = 0$, $\alpha_2 = 0$
B $t_3 = b$, $\alpha_3 = 2\pi$



Hence the Schwarzian transformation for this case is, in the general case of the three values a b c for t_1 t_2 t_3, respectively,

$$\frac{dz}{-C} = \frac{(t-a)\,(t-b)}{t-c}\,dt = \left((c-a) + (t-b) + \frac{(c-a)\,(c-b)}{t-c}\right) dt$$

and

$$\frac{z}{-C} = (c-a)\,t + \frac{(t-b)^2}{2} + (c-a)\,(c-b)\log\,(t-c) + \Gamma.$$

Putting $a = -1$, $b = b$, $c = 0$, we obtain

$$\frac{z}{-C} = t + \frac{(t-b)^2}{2} - b\log t + \Gamma.$$

If we make the point $t = -1$ correspond to the origin in the z-plane we shall have then for $t = -1$

$$o + o i = -1 + \frac{(b+1)^2}{2} - b \log(-1) + \Gamma.$$

Therefore the constant of integration is:

$$\Gamma = 1 - \frac{(b+1)^2}{2} + b \log(-1).$$

When t is very small the principal term is:

$$\frac{z}{C} = - b \log t + \cdot \cdot \cdot \qquad b \geqq 1.$$

Now as long as t is negative and its absolute value very small, the imaginary part must be equal to zero. But when t passes through the value zero from negative to positive, z increases by hi.

Hence $$z = - C\,(- b \log t - b \pi i)$$

which fulfils the above conditions,

and $$C b \pi = h \qquad C > 0 \text{ and real.}$$

$$C = \frac{h}{b \pi}$$

Also when $$t = b \qquad z = g + hi,$$
then

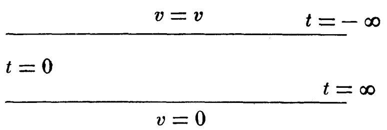
$$g + hi = - C\left(b - b \log b + 1 - \frac{(b+1)^2}{2} - b \pi i\right);$$

$$h = C b \pi$$

again, and which gives:

$$g = - C\left(b\,(1 - \log b) + 1 - \frac{(b+1)^2}{2}\right).$$

In the w-plane the diagram is:



and the corresponding transformation is, since the polygon has one zero angle at $t = 0$,

$$\frac{dw}{D} = \frac{dt}{t}. \qquad \therefore\ w = D \log t = \phi + i \psi.$$

When t is positive, $\psi = 0$;

when t is negative, $\psi = v$.

$$\therefore \quad D i \pi = v i, \qquad \text{or} \qquad D = \frac{v}{\pi},$$

and the final expression is

$$w = \frac{v}{\pi} \log t. \tag{a}$$

Now to find the amount of electricity on the under side of the upper plate, from the edge ($t = b$) to a point so far in ($z = -x + h i$) that the charge is uniform, we have, since

$$q = \frac{1}{4\pi} (\phi_{t=P} - \phi_{t=b})$$

or using eq. (a)

$$q = \frac{v}{4\pi^2} (\log t_P - \log b).$$

When t is very small and positive

$$z = -x + hi = -C\left(\frac{b^2}{2} - b \log t_P + 1 - \frac{(b+1)^2}{2} - b \pi i\right),$$

$$-x = -C\left(\frac{1-2b}{2} - b \log t_P\right)$$

$$\therefore \quad \log t_P = \frac{-x}{Cb} + \frac{1-2b}{2b},$$

from which

$$q = \frac{v}{4\pi^2}\left(\frac{-x}{Cb} + \frac{1-2b}{2b} - \log b\right),$$

and since $C = \dfrac{-h}{b\pi}$,

$$= \frac{-v}{4\pi h}\left[x + \frac{h}{\pi}\left(\frac{2b-1}{2b} + \log b\right)\right].$$

When b is unity, which brings the two edges symmetrically above one another, the correction becomes

$$q = \frac{-v}{4\pi h}\left(x + \frac{h}{2\pi}\right).$$

When t is very small and negative, $z = -x + o i$, and the quantity of electricity on the upper side of the lower plate is

$$q = \frac{-1}{4\,\pi}\,(\phi_{t=P} - \phi_{t=-1}) = \frac{v}{4\,\pi^2}\log(-\,t_P).$$

$$\therefore \quad q = \frac{-\,v}{4\,\pi^2}\left(\frac{-\,x}{C\,b} + \frac{1-2\,b}{2\,b}\right)$$

$$= \frac{+\,v}{4\,\pi\,h}\left[x + \frac{h}{\pi}\left(\frac{2\,b-1}{2\,b}\right)\right].$$

If b is infinite this becomes

$$q = \frac{+\,v}{4\,\pi\,h}\left(x + \frac{h}{\pi}\right).$$

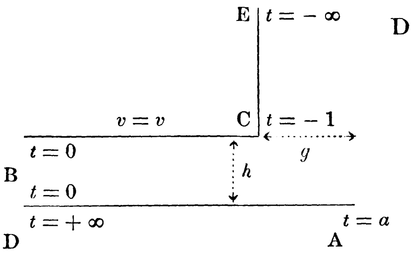
If b is unity the equation becomes

$$q = \frac{v}{4\,\pi\,h}\left(x + \frac{h}{2\,\pi}\right),$$

which agree with the known special cases already worked out.

Problem II.

Consider the problem of a square edge facing a semi-infinite plane, which may extend beyond the edge as per diagram.



The polygon consists of the sides DABCE.

At A	$t_1 = a,$	$\alpha_1 = 2\,\pi.$
B	$t_2 = c,$	$\alpha_2 = 0.$
C	$t_3 = b,$	$\alpha_3 = \frac{3\,\pi}{2},$

The transformation for this case is

$$\frac{1}{C}\frac{dz}{dt} = \frac{\sqrt{t-b}\,(t-b)}{t-c} = \sqrt{t-b}\left(1 + \frac{(c-a)}{t-c}\right),$$

which integrates into

$$\frac{z}{C} = \tfrac{2}{3}(t-b)^{\frac{3}{2}} + 2\,(c-a)\sqrt{t-b} + (c-a)\sqrt{c-b}\log\frac{\sqrt{t-b}-\sqrt{c-b}}{\sqrt{t-b}+\sqrt{c-b}} + \Gamma.$$

Let us put $a = a$, $b = -1$, $c = 0$; then

$$\frac{z}{C} = \tfrac{2}{3}\,(t+1)^{\frac{3}{2}} - 2\,a\sqrt{t+1} + a\log\frac{\sqrt{t+1}+1}{\sqrt{t+1}-1} + \Gamma.$$

Let the origin be at $t = -1$. Then

$$\frac{o + o\,i}{C} = a \log \frac{1}{-1} + \text{const.} \qquad \therefore \ \text{const} = \Gamma = a\,\pi\,i.$$

When $t = a$, $z = g + h\,i$; hence

$$g + h\,i = C\left(\tfrac{2}{3}\,(a+1)^{\frac{3}{2}} - 2\,a\sqrt{a+1} + a \log \frac{\sqrt{a+1}+1}{\sqrt{a+1}-1} + a\,\pi\,i\right).$$

$$h = C\,a\,\pi, \qquad C = \frac{h}{a\,\pi}.$$

From which

$$\text{(d)} \qquad g = \frac{h}{a\,\pi}\left(\tfrac{2}{3}(a+1)^{\frac{3}{2}} - 2\,a\sqrt{a+1} + a \log \frac{\sqrt{a+1}+1}{\sqrt{a+1}-1}\right).$$

The diagram in the w-plane consists of two infinite lines with one zero angle, the same as in the preceding problem.

The transformation for which is

$$\frac{dw}{D} = \frac{dt}{t}, \quad \text{or} \quad w = D \log t.$$

When t is negative, $\psi = v$. $\therefore$ $v = D\,\pi$, and $w = \frac{v}{\pi} \log t$.

The equation for the electricity on the lower face of the square corner from the corner $t = -1$ to a point so far in, that the distribution may be considered uniform is

$$q = \frac{v}{4\,\pi^2}\left(\log t_P - \log(-1)\right)$$

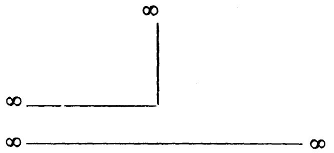
We must now find a value of t or of $\log t$ for a point at which t is negative and very small.

Putting $t = 0$ except in the terms which become large under such circumstances: —

$$-x + o\,i = C\left(\tfrac{2}{3} - 2\,a + 2\,a \log 2 - a \log t + a\,\pi\,i\right)$$

$$\log t = \frac{x}{a\,C} - \frac{2}{3\,a} + 2\,(1 - \log 2) - \pi\,i$$

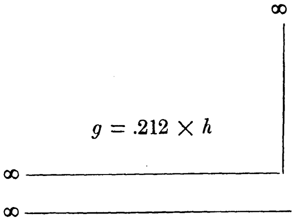
$$\therefore \quad q = \frac{v}{4\,\pi\,h}\left[x + \frac{2\,h}{\pi}\left(1 - \log 2 - \frac{1}{3\,a}\right)\right]$$



When a is infinite we have the case which agrees with that already worked out.

It is seen that when a is in the vicinity of unity the correction undergoes large variations. The correction becomes zero when:

$$1 = \log 2 + \frac{1}{3a} \quad \text{or } a = 1.086$$

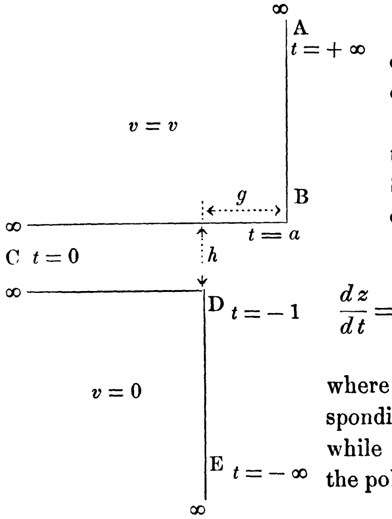


The value of g corresponding is found by substituting in formula (d) this value of (a); which makes

$$g = .212 \times h$$

or approximately as drawn.

PROBLEM III.



Find the capacity per unit length of the following system of conductors.

The dimensions perpendicular to the paper are all infinite. The Schwarzian transformation for this case is:

$$\frac{dz}{dt} = C(t-t_1)^{\frac{\alpha_1}{\pi}-1}(t-t_2)^{\frac{\alpha_2}{\pi}-1}(t-t_3)^{\frac{\alpha_3}{\pi}-1},$$

where t_1, t_2, t_3 are the values of t, corresponding to the three corners D, C, and B, while α_1, α_2, α_3 are the internal angles of the polygon EDCBA — E at D, C, and B.

$$\begin{array}{lll} \text{At D let } t_1 = b & \alpha_1 = \frac{3\pi}{2} \\ \text{C} \quad t_2 = c & \alpha_2 = 0 \\ \text{B} \quad t_3 = a & \alpha_3 = \frac{3\pi}{2} \end{array}$$

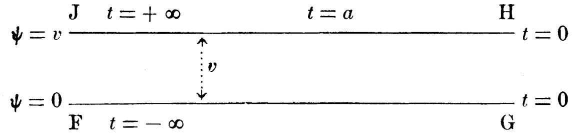
$$\therefore \quad \frac{dz}{dt} = \frac{C\sqrt{(t-a)(t-b)}}{t-c}$$

By making the substitution $u^2 = \frac{t-b}{t-a}$ or $t = \frac{a u^2 - b}{u^2 - 1}$ this expression becomes rationalized, and integrates into

$$z = C\left(\sqrt{(t-a)(t-b)} + \frac{2c-a-b}{2}\log\frac{\sqrt{t-a}+\sqrt{t-b}}{\sqrt{t-a}-\sqrt{t-b}}\right.$$

$$\left. -\sqrt{(b-c)(a-c)}\log\frac{\sqrt{(c-b)(t-a)}+\sqrt{(c-a)(t-b)}}{\sqrt{(c-b)(t-a)}-\sqrt{(c-a)(t-b)}} + \Gamma\right) \quad \text{(a)}$$

where Γ is the constant of integration.



The w-diagram consists of two straight lines parallel to the ϕ-axis, at a distance v apart.

At G $t = 0$, $\alpha = 0$, and the transforming function is obtained from $\frac{dw}{dt} = B\frac{1}{t}$ where B is a constant.

From this equation we obtain the integrated form

$$w = B(\log t + \text{const.}) = \phi + i\psi$$

As t runs from $-\infty$ to 0 and from 0 to $+\infty$, the value of $\log t$ diminishes by $i\pi$ as t passes through zero.

$$\therefore \quad w = B(\log t - i\pi) = \phi + i\psi$$

and as t passes through zero w increases by iv

$$\therefore \quad v = -B\pi \quad \text{or } B = -\frac{v}{\pi}.$$

Hence finally

$$w = \phi + i\psi = \frac{v}{\pi}(i\pi - \log t). \qquad \text{(b)}$$

Take now in equation (a)

$$\begin{aligned} a &= a > 1 \\ b &= -1 \\ c &= 0 \end{aligned}$$

which becomes

$$z = C\left(\sqrt{(t-a)(t+1)} + \frac{1-a}{2}\log\frac{\sqrt{t-a}+\sqrt{t+1}}{\sqrt{t-a}-\sqrt{t+1}}\right.$$
$$\left. - i\sqrt{a}\log\frac{\sqrt{t-a}+i\sqrt{a(t+1)}}{\sqrt{t-a}-i\sqrt{a(t+1)}} + \Gamma\right). \qquad \text{(c)}$$

Make the origin in the z-plane correspond to $t = -1$ and let $C = -iD$, then

$$z = o + oi = -iD\left(\frac{1-a}{2}\log 1 - i\sqrt{a}\log 1 + \Gamma\right)$$

from which follows $\Gamma = 0$.

When $\qquad t = a \qquad z = g + hi,$

then

$$z = g + hi = -Di\left(\frac{1-a}{2}\log(-1) - i\sqrt{a}\log(-1)\right)$$
$$= -Di\left(\frac{1-a}{2}\pi i - i\sqrt{a}\,\pi i\right).$$

From which follows, equating real and imaginary parts,

$$\begin{aligned} h &= -D\sqrt{a}\,\pi \\ g &= +D\frac{1-a}{2}\pi \end{aligned} \qquad \text{and } D = \frac{-h}{\sqrt{a}\,\pi}.$$

Equation (c) now becomes:

$$z = \frac{h}{\sqrt{a}\,\pi}\,i\left(\sqrt{(t-a)\,(t+1)} + \frac{1-a}{2}\log\frac{\sqrt{t-a}+\sqrt{t+1}}{\sqrt{t-a}-\sqrt{t+1}}\right.$$

$$\left. + \, i\sqrt{a}\log\frac{\sqrt{t-a}+i\sqrt{a\,(t+1)}}{\sqrt{t-a}-i\sqrt{a\,(t+1)}}\right). \qquad \text{(d)}$$

From the values for h and g above we find, if we put

$$\frac{g}{h} = b = \frac{a-1}{2\sqrt{a}}$$

$$a^2 - 2\,a\,(1 + 2\,b^2) + 1 = 0$$

whose solution is

$$a = 1 + 2\,b^2 \pm 2\,b\sqrt{1+b^2}.$$

We should expect *a priori* that a should be a function of $b = \frac{g}{h}$, as it is their ratio alone which determines the shape and relative position of the conductors. Since a must be greater than 1, we must take the larger root.

Suppose we take a point on one of the conductors so near to the value $t = 0$ that we may neglect t in comparison with 1 and *a fortiori* with a. We obtain by suitable transformations of the logarithms using the formula

$$\log\frac{x+iy}{x-iy} = 2\,i\tan^{-1}\frac{y}{x} + 2\,n\,\pi\,i.$$

Hence in formula (d), neglecting t and retaining 1 and a, we obtain

$$z = \frac{h}{\sqrt{a}\,\pi}\left(-\sqrt{a} + i\,\frac{1-a}{2}\,(2\,i\tan^{-1}\sqrt{a} - i\,\pi)\right.$$

$$\left. + \sqrt{a}\,(\log\frac{4\,a}{a+1} + i\,\pi + \log t_P)\right).$$

The imaginary part of this expression is $i\,y = \frac{h}{\sqrt{a}\,\pi}\sqrt{a}\,i\,\pi = i\,h$, which is a true result.

The real part is

$$-x = \frac{h}{\sqrt{a}\,\pi}\Big[-\sqrt{a} - (1-a)\left(\tan^{-1}\sqrt{a} - \frac{\pi}{2}\right)$$

$$+ \sqrt{a}\log\frac{4\,a}{a-1} + \sqrt{a}\log t_{P_0}\Big]$$

and

$$\log t_{P_0} = \frac{-x\,\pi}{h} - \left[1 + \frac{1-a}{\sqrt{a}}\left(\tan^{-1}\sqrt{a} - \frac{\pi}{2}\right) + \log\frac{a+1}{4\,a}\right].$$

Substituting this value of log t_P in the expression for the quantity of electricity on the conductor,

$$q = \frac{v}{4\,\pi^2}\Big(\log t_P - \log(-1)\Big)$$

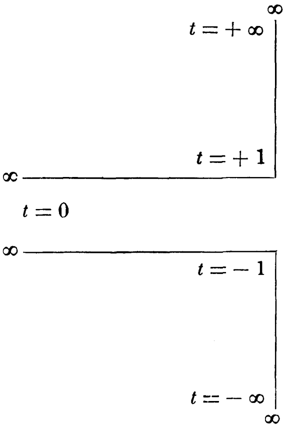
from $t = -1$ to $t = P$, we obtain the final expression for that quantity

$$\text{(e)}\quad q = -\frac{v}{4\,\pi\,h}\left\{x + \frac{h}{\pi}\left[\frac{1-a}{\sqrt{a}}\left(\tan^{-1}\sqrt{a} - \frac{\pi}{2}\right) + 1 - \log\frac{4\,a}{a+1}\right]\right\}.$$

When $v = 1$ this expression is the capacity of a strip of unit depth, from the point D to a point P_0 at a distance x towards the left; and we see that there is a correction of amount

$$x_c = \frac{h}{\pi}\left[\frac{1-a}{\sqrt{a}}\left(\tan^{-1}\sqrt{a} - \frac{\pi}{2}\right) + 1 - \log\frac{4\,a}{a+1}\right]$$

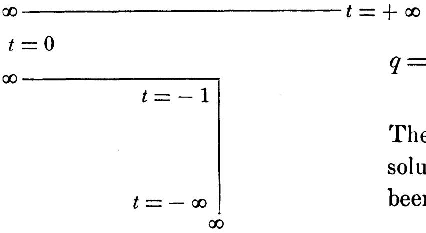
to be added to x, to take into account the concentration of the electricity at the corner. We may now calculate the amount of electricity on the face $D\,P$, assuming the distribution to be uniform, only instead of x we must use the value $x + x_c$.



When $a = 1$, or when we consider the case of two square corners facing one another, as in the figure, the equation (e) reduces to

$$q = \frac{-v}{4\pi h}\left(x + \frac{h}{\pi}(1 - \log 2)\right).$$

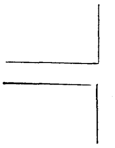
When we let a become infinite the expression (e) becomes indeterminate, but by determining its true value we find for the case of a square corner facing an infinite plane, the expression



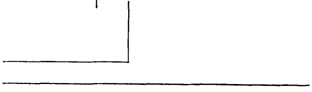
$$q = -\frac{v}{4\pi h}\left(x + \frac{2h}{\pi}(1 - \log 2)\right).$$

These two results check the general solution, as these special cases have been worked out by J. J. Thomson.*

As an approximate expression for these cases we may write:

$$q = -\frac{v}{4\pi h}\left(x + \frac{h}{10}\right)$$ for  I

and

$$q = -\frac{v}{4\pi h}\left(x + \frac{h}{5}\right)$$ for  II

where $\frac{h}{10}$ is used in place of $h \times .0978$.

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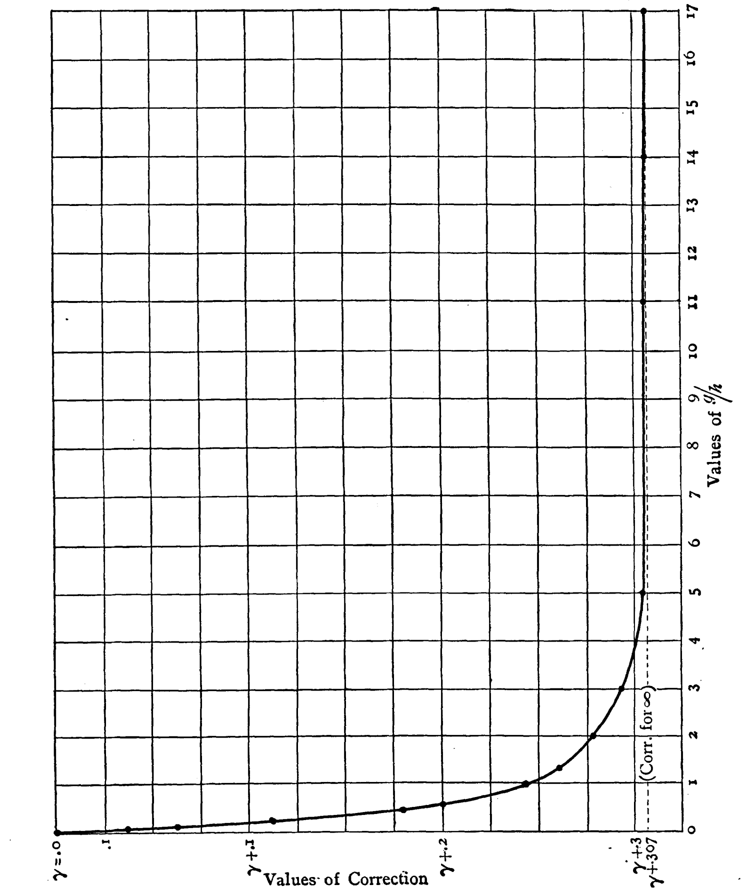
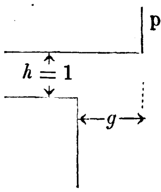
* Recent Researches in El. & Mag., Chapt. on Conjugate Functions.





Curve showing relation between the correction factor and g, the projection of the larger corner from the smaller.

The case where $g = 0$ has for correction factor $\frac{h}{\pi}(1 - \log 2)$

The case where $g = \infty$ has for correction factor $\frac{2h}{\pi}(1 - \log 2)$

The general case is

$$\frac{h}{\pi}\left\{\frac{a-1}{\sqrt{a}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{a}\right) + 1 - \log\frac{4a}{a+1}\right\}$$

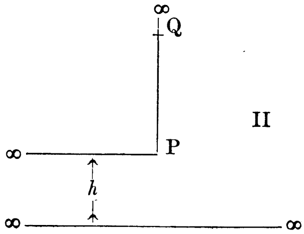
$$= \frac{h}{\pi}\left\{\underbrace{\frac{a-1}{\sqrt{a}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{a}\right) - \log\frac{2a}{a+1}} + (1 - \log 2)\right\}$$

where $\frac{g}{h} = \frac{a-1}{2\sqrt{a}}$ and the quantities plotted are $\frac{g}{h}$ and $\frac{a-1}{\sqrt{a}}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{a}\right) - \log\frac{2a}{a+1}$.

Looking at the plot, we notice the remarkable rapidity with which the projecting edge acts as if it were an infinite plane. In fact, for all purposes a projection of ten or twenty times the distance apart acts as nearly as possible as an infinite plane, as far as the electricity on the inside face is concerned.

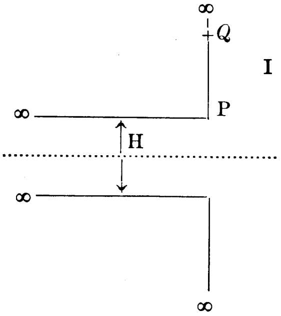
J. J. Thomson has found for the amount of electricity on the side PQ the expression:

$$q_{PQ} = \frac{v}{2\,\pi^2} \log \left(\frac{\pi\, PQ}{2\,h}\right).$$



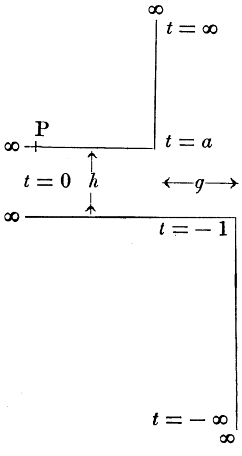
By the method of electrical images it is easily seen that for the case I, the expression is:

$$q = \frac{v}{\pi^2} \log \left(\frac{\pi\, Pq}{h}\right).$$



This problem was also completely solved with the following coördinates for the corners, where $a \leqq 1$. The solution for the quantity of electricity on the smaller plate being

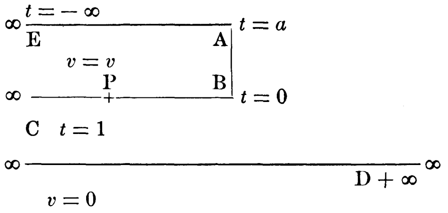
$$q = \frac{-\,v}{4\,\pi\,h} \left\{ \kappa + \frac{h}{\pi} \left[\frac{1-a}{\sqrt{a}} \left(\tan^{-1} \frac{\pi}{2} + \frac{\pi}{2} \right) \right.\right.$$
$$\left.\left. + 1 - \log \frac{4}{a+1} \right] \right\}.$$



This reduces to the correct expression for the case $a = 1$; but to obtain that for a rectangular corner near an infinite plane it cannot be applied, hence the change to the above notation.

Problem IV.

To find the distribution at the edge of a plate of finite width, which is at potential v, near to and parallel to an infinite plate at potential zero.



For generality put at

$$\text{A} \quad t_1 = a, \qquad \alpha_1 = \frac{3\pi}{2},$$

$$\text{B} \quad t_2 = b, \qquad \alpha_2 = \frac{3\pi}{2},$$

$$\text{C} \quad t_3 = c, \qquad \alpha_3 = 0.$$

The Schwarzian transformation then becomes:

$$\frac{dz}{C} = \frac{\sqrt{(t-a)\,(t-c)}}{t-c}\,dt. \tag{a}$$

When $a = b$ this assumes the form

$$\frac{dz}{c} = \frac{t-a}{t-c}\,dt = \left(1 + \frac{c-a}{t-c}\right)dt.$$

$$\therefore \quad \frac{z}{c} = \left[t + (c-a)\log(t-c) + \text{const.}\right]. \tag{b}$$

To integrate the above expression (a) make the substitution $t - c = v$,

then $\quad t - a = v + c - a = v + d, \qquad d = c - a,$

$\qquad\quad t - b = v + c - b = v + e, \qquad e = c - b,$

and $\quad d + e = 2c - (a + b), \qquad de = (c - a)\,(c - b)$

(a) now becomes

$$\frac{dz}{C} = \frac{\sqrt{(v+d)\,(v+e)}}{v}\,dv$$

and is of the form under the radical of

$$X = \alpha + \beta x + \gamma x^2 = de + (d + e)\,v + v^2,$$

where $\quad \alpha = de, \qquad \beta = d + e, \qquad \gamma = 1.$

Using Byerly's tables of integrals we find that

$$\frac{z}{C}=\underset{(147)}{\int\frac{\sqrt{X}}{x}dx}=\sqrt{X}+\frac{d+e}{2}\underset{(120)}{\int\frac{dx}{\sqrt{X}}}+de\underset{(142)}{\int\frac{dx}{x\sqrt{X}}}$$

where the numbers refer to the formulae used. In the use of (142) the product de or $(c-a)\,(c-b)$ must be positive; this is true, as a and b both lie on the same side of a and the differences are either both positive or both negative. Hence we have

$$\begin{aligned}\frac{z}{C}&=\sqrt{X}+\frac{d+e}{2}\log\left(\sqrt{X}+x+\frac{d+e}{2}\right)\\&\qquad-\sqrt{de}\log\left(\frac{\sqrt{X}+\sqrt{de}}{v}+\frac{d+e}{2}\frac{1}{\sqrt{de}}\right)+\Gamma\\&=\sqrt{(t-a)(t-b)}+\frac{2c-(a+b)}{2}\log\left(\sqrt{(t-b)(t-a)}+t-c+\frac{2c-(a+b)}{2}\right)\\&\quad-\sqrt{(c-a)(c-b)}\log\left(\frac{\sqrt{(t-a)(t-b)}+\sqrt{(c-a)(c-b)}}{t-c}+\frac{2c-(a+b)}{2\sqrt{(c-a)(c-b)}}\right)+\Gamma.\end{aligned}$$

As a check on this formula, put $a=b$; we obtain

$$\frac{z}{C}=(t-a)+(c-a)\log(t-c)+\text{const.},$$

which is the same as special case (b).

To apply this to our problem, put $a=a,\ \ b=0,\ \ c=1$; then we have

$$\begin{aligned}\frac{z}{C}&=\sqrt{t\,(t-a)}+\frac{2-a}{2}\log\left(\sqrt{t\,(t-a)}+t-1+\frac{2-a}{2}\right)\\&\qquad-\sqrt{1-a}\log\left(\frac{\sqrt{t\,(t-a)}+\sqrt{1-a}}{t-1}+\frac{2-a}{2}\frac{1}{\sqrt{1-a}}\right)+\Gamma.\end{aligned}$$

Using when convenient $m=\sqrt{1-a}$, this equation may be transformed into

$$\begin{aligned}\frac{z}{C}&=\sqrt{t\,(t-a)}+\frac{m^2+1}{2}\log\frac{(\sqrt{t}+\sqrt{t-a})^2}{2}\\&\qquad-m\log\frac{(\sqrt{m^2t}+\sqrt{t-a})}{2\,m\,(t-1)}+\Gamma.\qquad\text{(c)}\end{aligned}$$

Let the origin in the z-plane be chosen at the point B, $t = 0$; now putting $t = 0$ in (c), we obtain

$$\frac{o + o\,i}{C} = \frac{m^2 + 1}{2} \log \frac{m^2 - 1}{2} - m \log \frac{m^2 - 1}{-2m} + \Gamma.$$

$$\therefore \quad \Gamma = -\left(\frac{m^2 + 1}{2} \log \frac{m^2 - 1}{2} - m \log \frac{m^2 - 1}{-2m}\right).$$

When $t = a$, $\quad z = o + di.$

$$\begin{aligned} o + di &= C\left(\frac{m^2 + 1}{2} \log \frac{1 - m^2}{2} - m \log \frac{1 - m^2}{-2\mathrm{m}}\right) \\ &- C\left(\frac{m^2 - 1}{2} \log \frac{1 - m^2}{-2} - m \log \frac{1 - m^2}{2m}\right) \\ &= C\left(\frac{m^2 + 1}{2} \pi i - \frac{2m\pi i}{2}\right) = \frac{C\pi i}{2}(m - 1)^2. \\ &\therefore \quad d = \frac{C\pi}{2}(m - 1)^2. \end{aligned}$$

To find h, we notice that where t is very small we may write

$$\frac{dz}{C} = \frac{\sqrt{t\,(t - a)}}{t - 1}\,dt \quad \text{as} \quad \sqrt{1 - a}\,\frac{dt}{t - 1} + \ldots$$

$$\therefore \quad \frac{z}{C} = m\,[\log\,(t - 1) + \text{const.}].$$

Const. $= -\,i\,\pi$, and is so chosen that the imaginary part of z is zero as long as t is less than unity.

$$\therefore \quad z = m\,C\,[\log\,(t - 1) - i\,\pi].$$

When $\qquad t = 1 + \epsilon, \qquad z = -\,\infty - i\,h;$

$$\therefore \quad -\,\infty - i\,h = C\,m\,[\log|\epsilon| - i\,\pi],$$

or $\qquad h = C\,m\,\pi, \qquad C = \dfrac{h}{m\,\pi}.$

Hence $\qquad \dfrac{d}{h} = \dfrac{(m - 1)^2}{2\,m} = K, \quad$ say $\quad m \gtreqless 1.$

From which, given any value of K which determines the configuration of the polygon, we can find m by the formula

$$m = 1 + K \pm \sqrt{K(K+2)},$$

and vice versa.

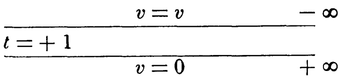
The diagram in the w-plane consists of two parallel straight lines, the angle being zero, and the value of t corresponding being $t = +1$.

Hence

$$\frac{dw}{B} = \frac{dt}{t-1},$$

$$w = \phi + i\,\psi = B[\log(t-1) + \text{const.}].$$

Const. $= 0$, because $\psi = 0$ as long as $t > 1$. When t passes through the value $t = 1$ and becomes less than unity, ψ increases by $v = B\pi$.



Hence $B = \dfrac{v}{\pi}$.

Now the quantity of electricity on a strip of breadth x and depth unity is by the general equation:

$$q = \frac{1}{4\pi}(\phi_t - \phi_0);$$

and since

$$w = \frac{v}{\pi}\log(t-1),$$

$$q = \frac{v}{4\pi^2}[\log(t-1) - \log(-1)]$$

$$= \frac{v}{4\pi^2}\log(1-t), \qquad \text{(c)}$$

which gives the amount of electricity on a strip of breadth x, on the under side of the upper plate from the corner B, to a point P at a distance $-x$, from B.

Let us now find a value of $\log(t-1)$ so far from the edge and on the lower surface of the upper plate that t may be considered nearly equal to unity, say $t = 1 - \epsilon$. From (a)

$$\frac{z}{C} = m + \frac{1+m^2}{2}\log\frac{(1+m)^2}{2} - m\log\frac{2m}{t-1}$$

$$-\frac{1+m^2}{2}\log\frac{m^2-1}{2} + m\log\frac{m^2-1}{-2m}$$

$$= m + \frac{1+m^2}{2}\log\frac{(m+1)^2}{m^2-1} - m\log\frac{4m^2}{1-m^2} + m\log(t-1);$$

$$\log(t-1) = \frac{z}{Cm} - 1 - \frac{1+m^2}{2m}\log\frac{m+1}{m-1} + \log\frac{4m^2}{1-m^2};$$

$$\log(1-t) = \frac{-x}{Cm} - 1 - \frac{1+m^2}{2m}\log\frac{m+1}{m-1} + \log\frac{4m^2}{m^2-1}.$$

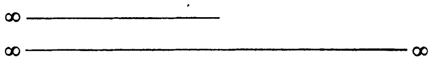
Substituting this value of log $(1-t)$ in (c) we obtain

$$q = -\frac{v}{4\pi h}\left[x + \frac{h}{\pi}\left(1 + \frac{1+m^2}{2m}\log\frac{m+1}{m-1} - \log\frac{4m^2}{m^2-1}\right)\right] \quad \text{(d)}$$

which is the required expression.

When $a = 0 \quad m = 1$

which gives the case of a semi-infinite plate near an infinite plate

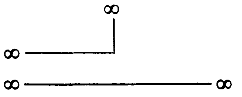


this case (d) reduces to

$$q = \frac{-v}{4\pi h}\left(x + \frac{h}{\pi}\right),$$

which is a correct result.

Where $a = -\infty \quad m = +\sqrt{\infty}$
this should give the expression for the electricity on a square corner near an infinite plate



(d) becomes:

$$q = \frac{-v}{4\pi h}\left[x + \frac{h}{\pi}\left(1 + \frac{\sqrt{\infty}}{2}\log 1 - \log 4\right)\right].$$

We must find the true value of the indefinite term

$$\frac{m^2+1}{2\,m}\log\frac{m+1}{m-1}.$$

Writing it in the form $\frac{0}{0}$ we have

$$\text{Limit } m \doteq \sqrt{\infty}\ \frac{\dfrac{d}{d\,m}\log\dfrac{m+1}{m-1}}{\dfrac{d}{d\,m}\quad\dfrac{2\,m}{m^2+1}} = \frac{(m^2+1)^2}{(m^2-1)^2}_{m\doteq\infty} = +\,1.$$

Hence

$$q = -\frac{v}{4\,\pi\,h}\left(x + \frac{2\,h}{\pi}\,(1-\log 2)\right)$$

which is the correct result required.

Since for a plate of no thickness the parenthesis of the correction is unity and for infinite thickness is 2 (1 — log 2) = .614.

The correction then always lies between the values: $\frac{h}{\pi}$ and $\frac{h}{\pi}$ .614 or $h \times .318$ and $h \times .195$.

The amount of electricity on the end $A\,B$ is given by:

$$q = \frac{1}{4\,\pi}\,(\phi_a - \phi_o) = \frac{v}{4\,\pi^2}\,[\log\,(a-1) - \log\,(-\,1)]$$

$$= \frac{v}{4\,\pi^2}\log\,(1-a) = \frac{v}{4\,\pi^2}\log m^2;$$

$$q = \frac{v}{2\,\pi^2}\log m.$$

Now the equation $m = 1 + K + \sqrt{K(K+2)}$ gives the value of m for any value of $K = \frac{d}{h}$, the ratio of the thickness of the upper plate to its distance from the lower. We must take the positive value of the radical as $m \geqq 1$.

When $d = 0$ or $m = 1$, the amount of electricity on the sharp edge is

$$q = \frac{v}{4\,\pi^2} \log 1 = 0$$

and we find there is none.

To find the quantity of electricity on the upper surface of the upper plate, we find the quantity which lies on the plate from the corner B, at which $t = 0$, to a point P on the upper surface such that the corresponding x is a large multiple of h.

We must, however, agree to keep a small so that we can neglect it in comparison with the value of t we choose. From the general equation (a) we have

$$\frac{z}{C} = t + \frac{m^2+1}{2} \log 2 \mid t \mid - m \log \frac{(m+1)^2}{2\,m}$$
$$- \frac{m^2+1}{2} \log \frac{m^2-1}{2} + m \log \frac{m^2-1}{-2\,m};$$

$$-\frac{x}{C} = t + \frac{m^2+1}{2} \log \mid t \mid + \frac{\mathrm{m}^2+1}{2} \log 4$$
$$- \frac{(m-1)^2}{2} \log (m-1) - \frac{(m+1)^2}{2} \log (m+1)$$

and very approximately since $\mid t \mid$ is large.

$$\therefore \quad 1 - t = 1 + \frac{x}{C} + \frac{m^2+1}{2} \log \left(1 + \frac{x}{C}\right) - \frac{(m-1)^2}{2} \log (m-1)$$
$$- \frac{(m+1)^2}{2} \log (m+1) + \frac{m^2+1}{2} \log 4 \qquad \text{(e)}$$

and $$\frac{1}{C} = \frac{\pi\, m}{h}.$$

The quantity of electricity on the upper plate from $t = 0$ to $t = P$ is

$$q = \frac{v}{4\,\pi^2} [\log (t-1) - i\,\pi] = \frac{v}{4\,\pi^2} \log (1-t).$$

$\therefore$ Using the expression (e) we have

$$q=\frac{v}{4\pi^2}\log\left[1+\frac{\pi x m}{h}+\frac{m^2+1}{2}\log\left(1+\frac{\pi m x}{h}\right)-\frac{(m-1)^2}{2}\log(m-1)\right.$$

$$\left.-\frac{(m+1)^2}{2}\log(m+1)+\frac{m^2+1}{2}\log 4\right]. \qquad \text{(g)}$$

The expression for $m=1$ or $a=0$ should reduce to the known expression. It becomes

$$q=\frac{v}{4\pi^2}\log\left[1+\frac{\pi\kappa}{h}+\log\left(1+\frac{\pi\kappa}{h}\right)\right]$$

which is identical with his result.